Crossing Probabilities and Modular Forms

Peter Kleban
University of Maine

Don Zagier
MPI für Mathematik, Bonn
Collège de France

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Materials Theory

- Percolation
  - the model
  - crossing probabilities
- Connection to the Potts models
- Boundary Conformal Field Theory

- Review of modular forms
- Modular properties of crossing (I)
- Crossing in SLEs
- Modular properties of crossing (II)
- Higher–order modular forms

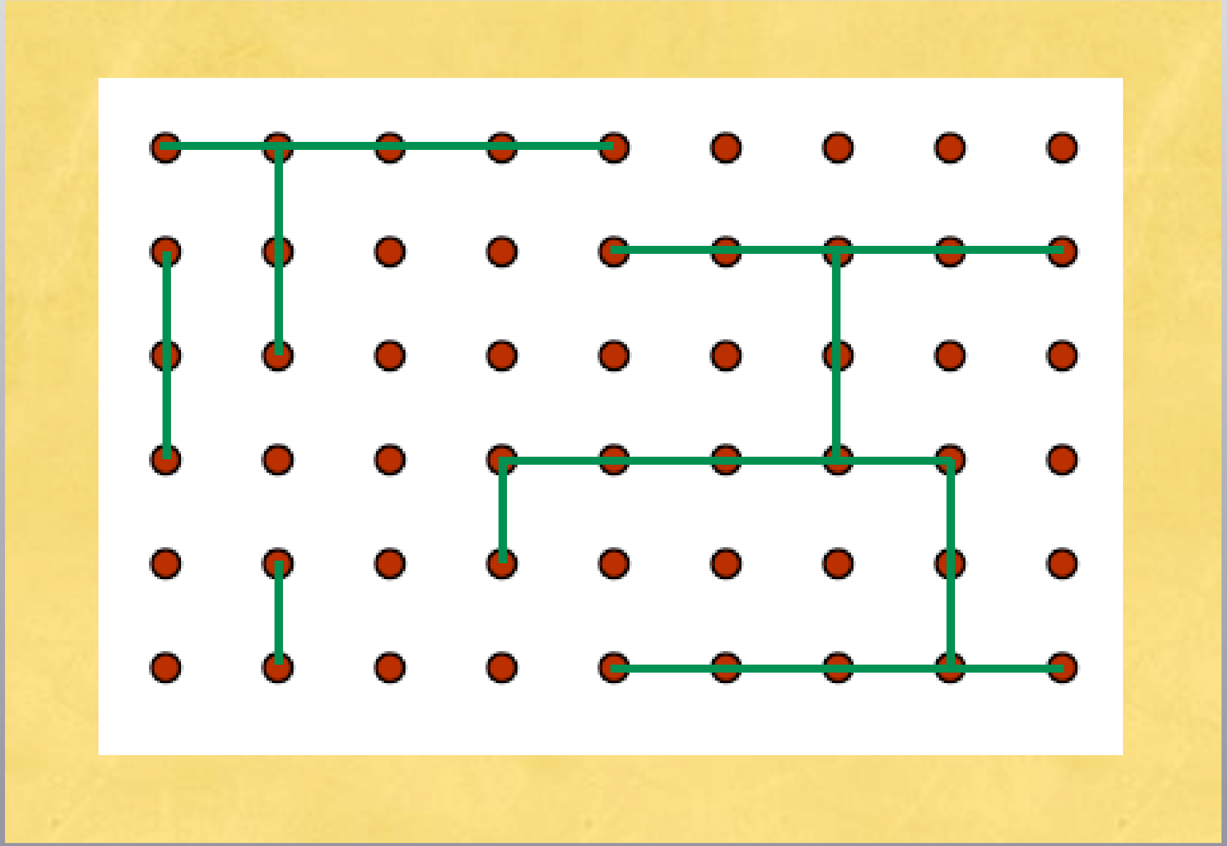
This is a tale of two symmetries–triangular symmetry (for percolation, at least), and modular symmetry–or, to be more precise, covariance.

The modular story also involves the interplay of two other symmetries–duality (TBD) and rotation through 90^0, which conspire to give rise to a new kind of modular object.

What arises, in the main, is a modular way to characterize the crossing probabilities (not quite a derivation).

Percolation

Percolation in 2-D is deceptively easy to define. Imagine a large square lattice of points, with bonds between neighboring points occupied with (independent) probability p. A given configuration might look like this:



As you can see, the occupied bonds form clusters. The geometric properties of these clusters are what we study. One can also have percolation in many other cases, for example on the sites of a hexagonal lattice (see below).

When p is small (near 0), the lattice will be mostly empty (for the great majority of configurations). When p is large (near 1), it will be mostly full. If we let the lattice get very large, there is rigorously known to be a phase transition (at p = 1/2 for the bond model shown). At this p value (the percolation point p_c), an infinite cluster can appear on the lattice. Therefore it will be possible to get from one side to the other along a cluster–there is a non-zero probability of crossing.

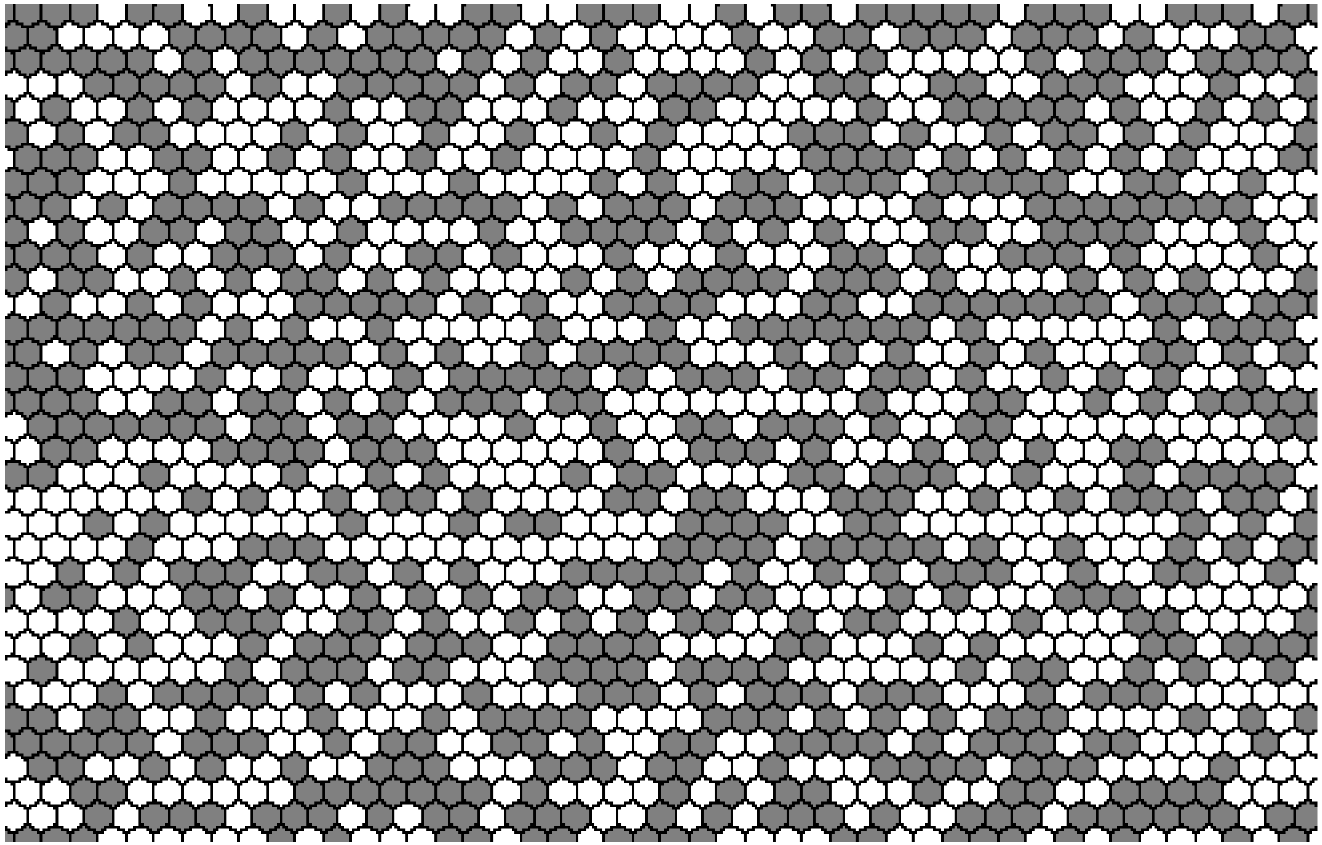
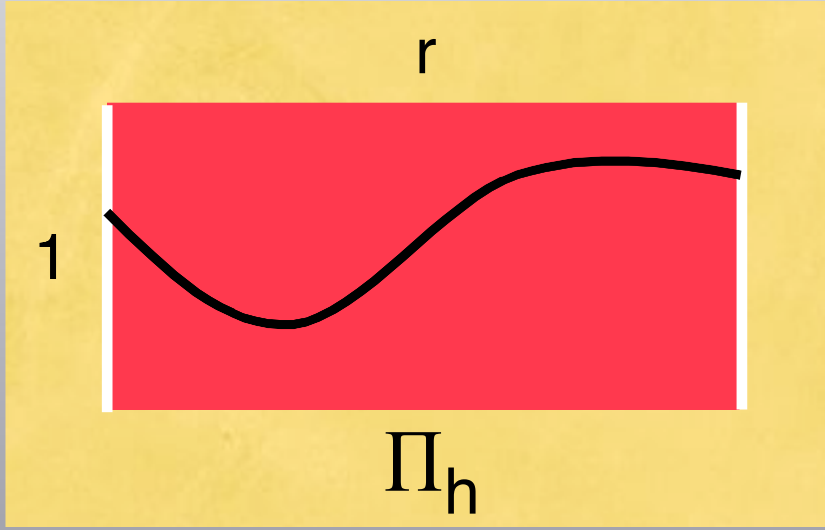
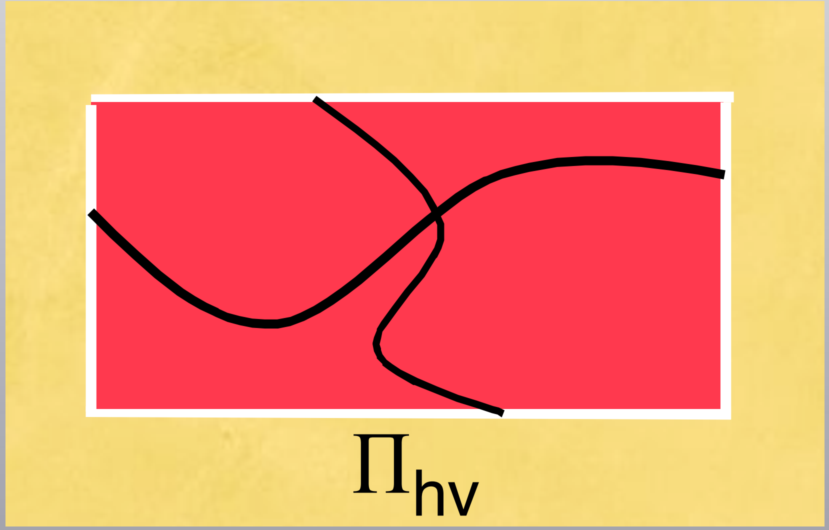


Fig. 10.1. Is there a left to right crossing of white hexagons?

The rest of this talk considers the percolation point only, since conformal field theory applies there. CFT gives us differential equations whose solutions describe various quantities, for example the crossing probability (or the density of a cluster).

We consider a large, rectangular lattice of aspect ratio r, and two types of crossing probabilities: $\Pi_h(r)$, the probability of a horizontal crossing, and $\Pi_{hv}(r)$, the probability of connecting all four sides of the rectangle.





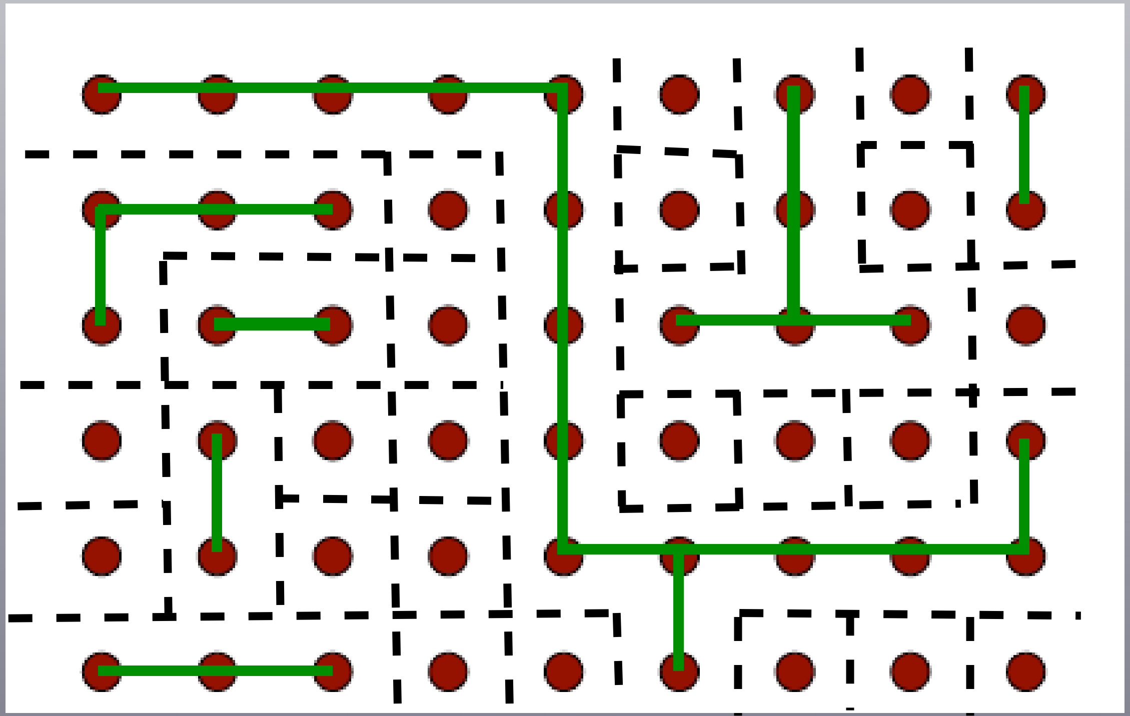
A consequence of the conformal invariance of this problem is that the crossing probabilities depend only on the aspect ratio r (for a large lattice, of course).

[Langlands, Saint-Aubin, Pichet, Pouliot]

The horizontal probability satisfies a symmetry called duality. Consider the bond percolation problem defined above. For each configuration, either:

(i) there is a horizontal crossing on the lattice, in which case the dual lattice has no vertical crossing, or

(ii) there is no horizontal crossing on the lattice and the dual lattice has a vertical crossing.



occupied lattice bond:

occupied dual lattice bond:

Now one or the other must occur. Therefore

$$\Pi_h(r) + \Pi_v(r) = 1,$$

where $\Pi_v(r)$ is the probability of a vertical crossing. But $\Pi_v(r) = \Pi_h(1/r)$, the probability of a horizontal crossing on the original lattice turned by 90^0.

Hence

$$\Pi_h(r) + \Pi_h(1/r) = 1.$$

On the other hand, by construction the horizontal–vertical crossing probability clearly satisfies

$$\Pi_{hv}(r) = \Pi_{hv}(1/r).$$

Differentiating we find

$$\Pi'_h(r) = +\,(1/r^2)\Pi'_h(1/r),$$

$$\Pi'_{hv}(r) = -(1/r^2)\Pi'_{hv}(1/r).$$

The difference between these two equations, the minus sign, plays a crucial role in their modular properties, and is responsible for the appearance of a new kind of modular object, as we will see.

The Potts Models

The derivation of the crossing formulas for percolation makes use of known CFT results for the Potts models. The connection is made through the mapping of Fortuin and Kastelyn. The Potts model is a generalization of the Ising model in which the spins s(r) at each site of a lattice take the values (1, 2, . . . ,Q), where, initially, Q is an integer (Q = 2 is the Ising model). The energy is the sum over all bonds (r′, r′′) between sites of $-J\,\delta_{s(r')s(r'')}$. Thus the partition function (from which the thermodynamics follows) is

$$Z = \mathrm{Tr}\,\exp(\beta J \sum_{r',r''} \delta_{s(r'),s(r'')})$$

Apart from an overall unimportant constant the partition function may be rewritten as

$$Z = \mathrm{Tr} \prod_{r',r''} \left((1-p) + p\delta_{s(r'),s(r'')} \right)$$

where $p = 1 - e^{-\beta J}$.

Now one expands the product. If there are B bonds on the lattice there will be 2^B terms in this expansion. In any term, each bond is open (if we choose the term ∝ p), or closed (if we choose the term ∝ (1 – p)). Sites connected by open bonds form clusters; the deltas force all the spins in a given cluster to be in the same state, so each cluster contributes a factor Q.

Thus we can write Z as a sum over configurations C of open bonds:

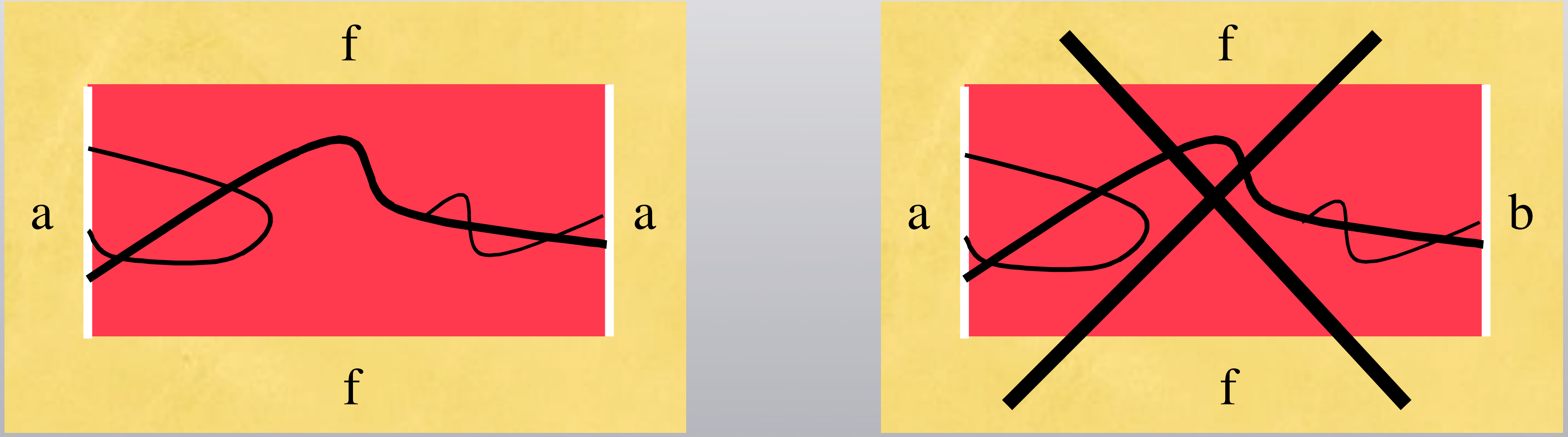
$$Z = \sum_{\mathcal{C}} p^{|\mathcal{C}|}(1-p)^{B-|\mathcal{C}|}\, Q^{N(\mathcal{C})}$$

where $|C|$ is the number of open bonds in the configuration and $N(C)$ is the number of distinct clusters in C. This is the random cluster representation of the Potts models.

When $Q = 1$, the sum is simply over all possible configurations weighted by their probabilities, which is bond percolation.

Now $Z(Q = 1) = 1$, but there is nontrivial information in the correlation functions. Here Z is a polynomial in Q so its definition may be extended to non-integer values of Q. The upshot is that percolation corresponds to the $Q \to 1$ limit of the Potts model.

How does this relate to crossing probabilities? The key idea is to consider boundary conditions. Suppose we have a Potts model on a rectangle with the spins on the boundary specified to all be in state Q = a on the left hand edge, Q = b on the right, and free (unconstrained) on the top and bottom.



Their are two cases of interest, a = b and a ≠ b, with corresponding partition functions $Z_{aa}(Q)$ and $Z_{ab}(Q)$. Now each configuration *C* has a cluster crossing from left to right or not. Those which do not cross contribute to both partition functions (with the appropriate weights), but those which do cross cannot contribute to $Z_{ab}(Q)$, since the boundary spins differ by assumption.

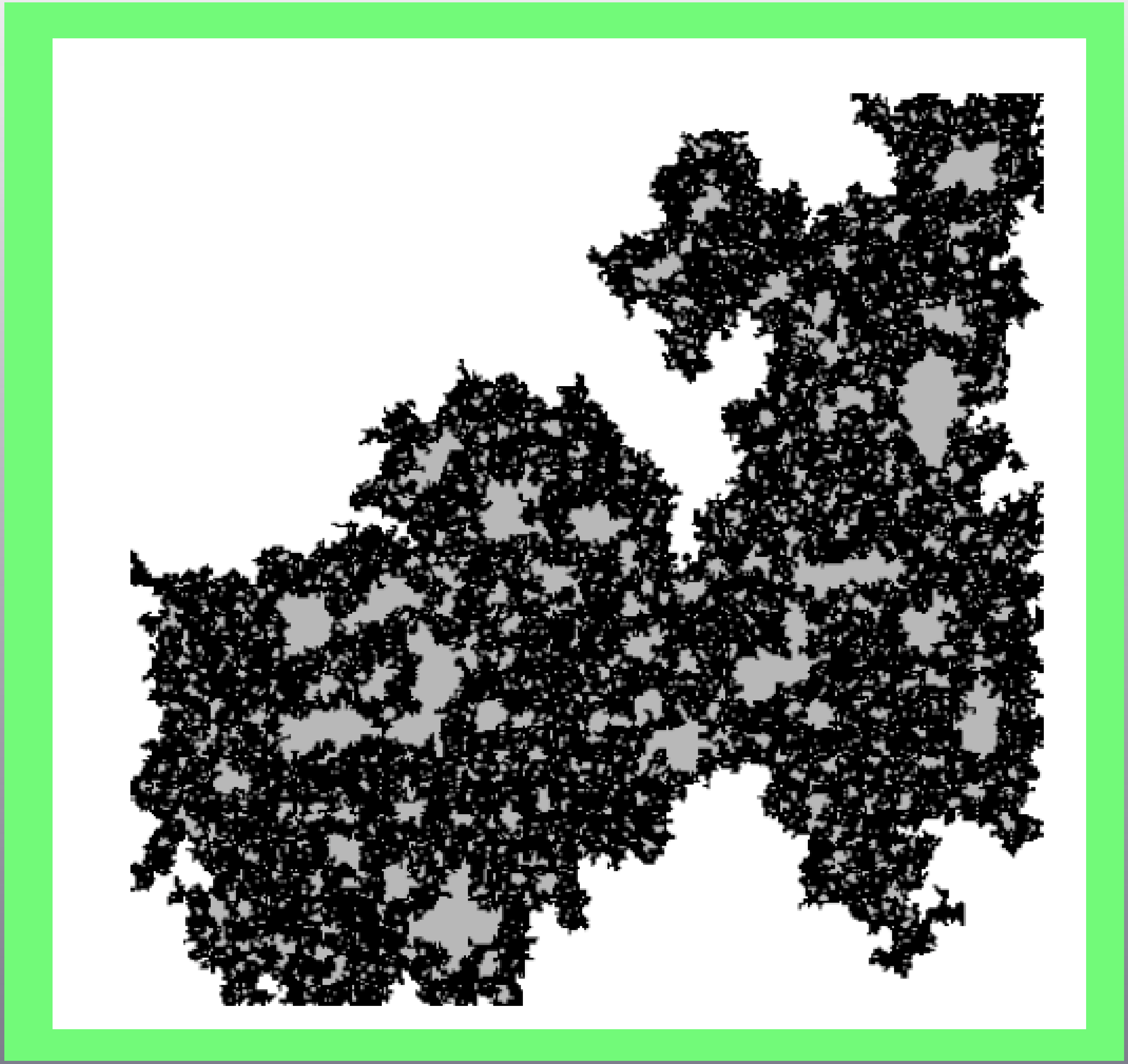
Therefore we have the important result

$$\Pi_h(r) = \lim_{Q \to 1} (Z_{aa}(Q) - Z_{ab}(Q)).$$

(Recall that each Z is a polynomial in Q, so $Z_{ab}(Q)$ makes sense even for $Q = 1$.) This equation is used to extrapolate known conformal results for the Potts models to percolation. The partition functions are expressed in terms of correlations of boundary conformal operators, as we will see...

[Cardy]

The clusters illustrations I've use are quite schematic. At the percolation point, clusters are ramified. Here is an example of (part of) such a cluster on a square:



Conformal Field Theory

CFT is a big topic; we only mention a few features. One of its main elements is the assumption (which should apply at many two–dimensional critical points) of the conformal transformation properties of (primary) operator correlation functions:

$$< \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\ldots >=$$

$$w'(z_1)^{h_1}\bar{w}'(\bar{z}_1)^{\bar{h}_1}w'(z_2)^{h_2}\bar{w}'(\bar{z}_2)^{\bar{h}_2}\ldots < \phi_1(w(z_1), \bar{w}(\bar{z}_1))\phi_2(w(z_2), \bar{w}(\bar{z}_2))\ldots >$$

Here, <...> is an expectation value, w(z) is an arbitrary conformal map, and $(h_i, \bar{h}_i)$ are the conformal weights (generally both real) of the operator $\phi_i(z, \bar{z})$.

[Belavin, Polyakov, Zamolodchikov]

It is intriguing to focus on the behavior of just one variable

$$< \phi(z,*)\ldots >= w'(z)^h \ldots < \phi(w(z),*)\ldots >$$

since this equation (in itself) is the same as the condition satisfied by modular forms (the allowed "w(z)" are not, of course, and modular forms satisfy additional conditions).

The main point of interest here is that a change of (appropriate) boundary conditions is implemented by conformal boundary operators. In particular, our partition function satisfies

$$Z_{ab} = Z_f < \phi_{fa}(z_1)\phi_{af}(z_2)\phi_{fb}(z_3)\phi_{bf}(z_4) >$$

Here Z_f is the partition function with free boundary conditions on all sides (so Z_f = 1 for percolation), and the z_i are the positions of the four corners of the rectangle. The operator $\phi_{ab} = \phi_{(1,3)}$ has been identified for $Q = 2$ (Ising model) and $Q = 3$.

[Cardy]

One then makes use of the operator product expansion (another important CFT tool) to find ϕ_{af}

$$\phi_{af}(z+\epsilon)\phi_{fb}(z-\epsilon) \sim I\ \delta_{a,b} + \phi_{ab}(z)$$

By using this equation and the known properties of CFT operators, it follows that $\phi_{af} = \phi_{(1,2)}$. This conclusion is buttressed by the fact that $h_{(1,2)} = 0$ for $Q = 1$ (percolation). The vanishing of this weight means that Z_{ab} is conformally invariant for percolation, in agreement with numerical simulations and the rigorous proof of Smirnov.

The upshot is that we need to determine the four–point correlation function

$$< \phi_{(1,2)}(z_1)\phi_{(1,2)}(z_2)\phi_{(1,2)}(z_3)\phi_{(1,2)}(z_4) >$$

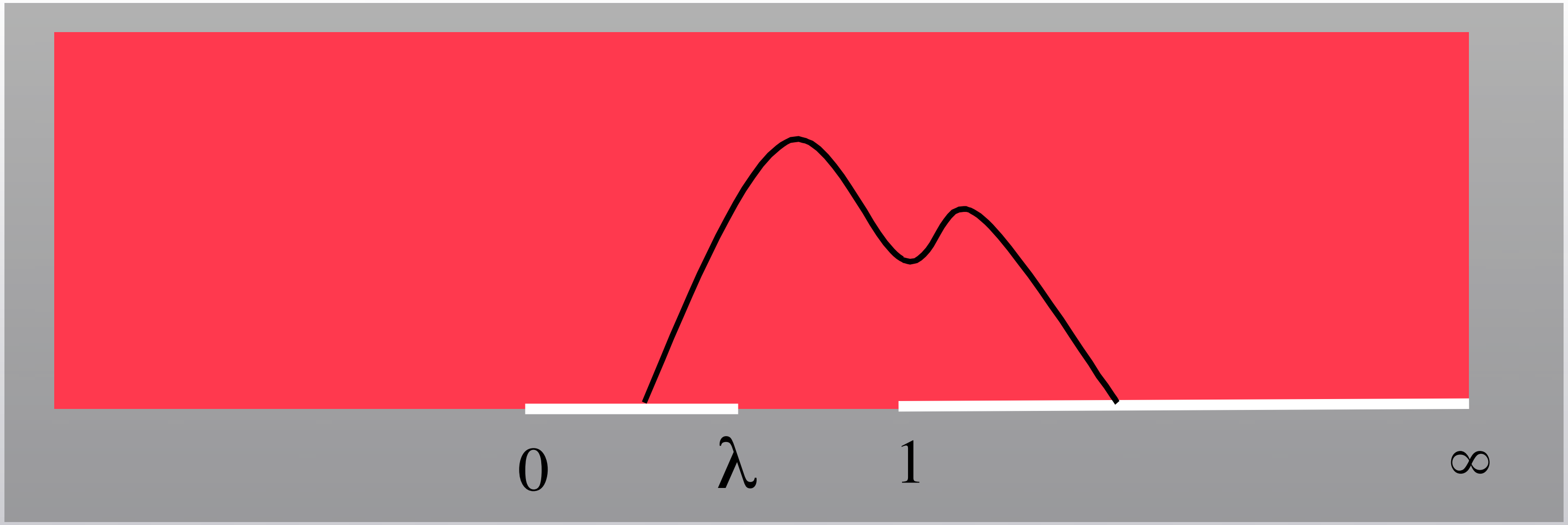
for percolation (which, for the afficionados, corresponds to central charge c = 0).

Generally, conformal invariance implies

$$< \phi_a(z_1)\phi_b(z_2)\phi_c(z_3)\phi_d(z_4) >= \frac{1}{(z_1 - z_2)^{h_a+h_b}(z_3 - z_4)^{h_c+h_d}} F(\lambda)$$

where λ is the cross–ratio of the four points. Here, all the h = 0, so only the factor *F*(λ) remains. Furthermore, for the $\phi_{(1,2)}$ operator, *F* satisfies a Riemann (second–order) differential equation.

It is conventional to chose the four points on the real axis, so that the crossing is in the upper half plane. We can choose the points as shown



The half–plane crossing probability $F(\lambda)$ then must satisfy $F(\lambda) \to 0$ as $\lambda \to 0$ (corresponding to $r \to \infty$ on the rectangle) and $F(\lambda) \to 1$ as $\lambda \to 1$ ($r \to 0$). The differential equation has two solutions, one of which is a constant. Choosing the correct linear combination then gives Cardy's formula for the rectangle

$$\Pi_h(r) = \frac{2\pi\sqrt{3}}{\Gamma(1/3)^3} \lambda^{1/3} \; {}_2F_1(1/3, 2/3; 4/3; \lambda).$$

(Here the aspect ratio r is expressed as a function of the cross–ratio λ by means of the conformal map from the half–plane to the rectangle.)

(The exponent 1/3 which appears is the conformal weight $h_{(1,3)}$ of the ϕ_{ab} operator for percolation.) The hypergeometric function is special (satisfying c – a = 1) which means it may be written as

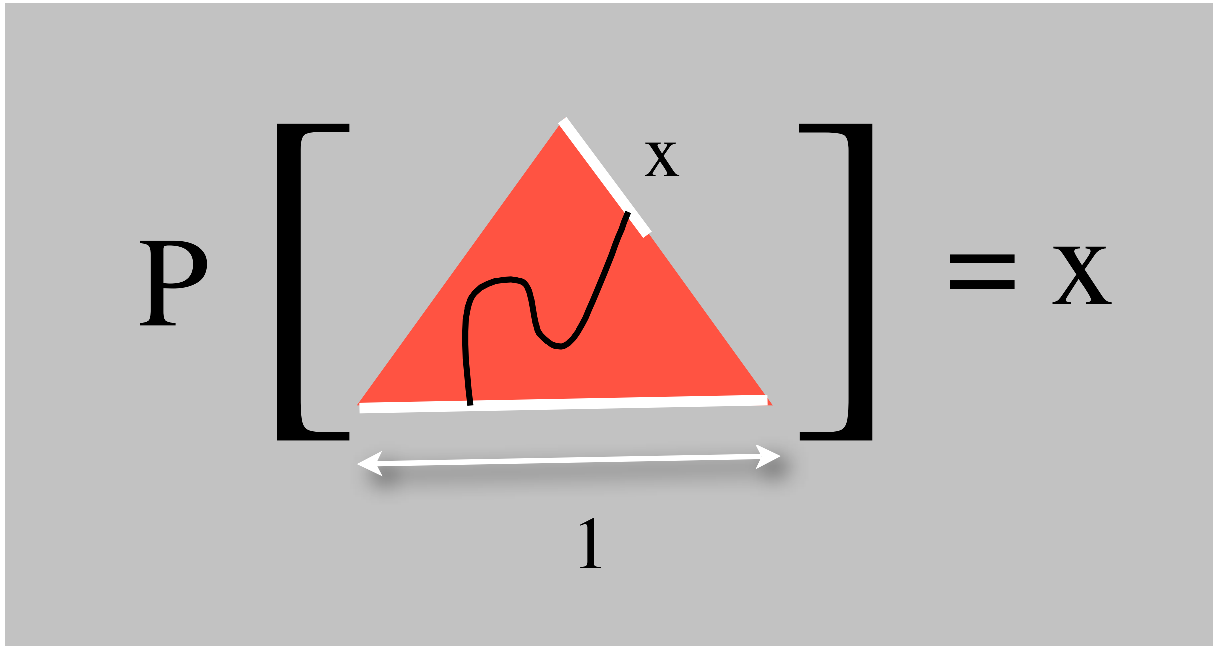
$$\Pi_h(r) = \frac{2\pi}{\sqrt{3}\Gamma(1/3)^3} \int_0^\lambda (t(1-t))^{-2/3} dt.$$

This reflects the fact that the hypergeometric equation satisfied by *F* factors as

$$\frac{d}{d\lambda}(\lambda(1-\lambda))^{2/3}\frac{d}{d\lambda}F = 0.$$

Now the integral form for *F* is in fact a Schwarz–Christoffel mapping. Carleson noticed this and pointed out...

that as a result, on an equilateral triangle, Cardy's formula implies



This striking result illustrates the triangular symmetry of percolation(regardless of the underlying lattice symmetry). Notice that the conformal weight 1/3 has become the angle of the triangle.

A similar conformal analysis for the horizontal–vertical crossing probability has been carried out by Watts. In that case the crossing satisfies a fifth–order differential equation, which may be written as

$$\left(\frac{d}{d\lambda}\lambda^{-1}(1-\lambda)^3\frac{d}{d\lambda}\lambda^2\right)\left(\frac{d}{d\lambda}(\lambda(1-\lambda))^{1/3}\frac{d}{d\lambda}(\lambda(1-\lambda))^{2/3}\frac{d}{d\lambda}\right)F \;=\; 0\,.$$

Setting the rightmost factor equal zero gives rise to the solution set 1, $\Pi_h(r)$, and $\Pi_{hv}(r)$. The probability of a horizontal crossing with no vertical crossing, $\Pi_{h\underline{v}}(r) := \Pi_h(r) - \Pi_{hv}(r)$ again has a simple form. It is expressible in terms of a hypergeometric function $_3F_2$ or as an integral

$$\Pi_{hv}(r) = \frac{1}{\sqrt{3}\pi}\int_0^\lambda (t(1-t))^{-2/3}\int_0^t (u(1-u))^{-1/3}du\,dt\,.$$

This function (or more precisely, its derivative) is of special interest, since it is a “higher order” modular form, as we will see. (Note recent rigorous proof by Dubédat.)

Next we need to write the cross–ratio λ in terms of the aspect ratio r. Letting τ = ir, one finds that λ(τ) is the classical modular function ("Hauptmodul") for the subgroup Γ(2) of *PSL*(2,Z). This may be expressed, for instance, by

$$\lambda(\tau) \;=\; 16\frac{\eta(\tau/2)^8\eta(2\tau)^{16}}{\eta(\tau)^{24}}$$

where η(τ) is the Dedekind η–function (which is a modular form of weight 1/2). Now define, as usual, $q = e^{2\pi i \tau}$. As r → ∞, λ(ir) → √q := $\hat{q}$

The appearance of $\hat{q}$ is ubiquitous in CFT on a rectangle, and we will see it again below (especially with regard to the theta group).

It is convenient to work with the r–derivatives of the crossing probabilities. One finds, with $f_1 \propto \Pi'_h(r)$ and $f_2 \propto \Pi'_{h\underline{v}}(r)$

$$f_1(\tau) = \eta(\tau)^4 \,,$$
$$f_2(\tau) = -\frac{2\pi i}{3}\,\eta(\tau)^4 \int_\tau^\infty \frac{\eta(z/2)^8 \eta(2z)^8}{\eta(z)^{12}}\, dz \,.$$

The function f_1 is a modular form of weight 2, but f_2 is a new type of modular object, as we will see...

A "Crash Course" in Modular Forms

Modular forms can be defined as holomorphic functions f(τ), with τ in the upper half–plane, that have certain transformation properties under the (full) modular group Γ_1. This is the subset of the Möbius transformations given by

$$\gamma(\tau) = \frac{a\tau + b}{c\tau + d}$$

$$a, b, c, d \in \mathbf{Z};\ ad - bc = 1.$$

The defining property of a modular form of weight k (generally an integer) is

$$f(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^k f(\tau).$$

(To follow up on our previous remark, note that this may be written as

$$f(\tau) = \gamma'(\tau)^{k/2} f(\gamma(\tau).)$$

Now Γ_1 may be generated by the operations

T: $\tau \to \tau+1$

S: $\tau \to -1/\tau$.

These are implemented by the matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Hence we can replace the condition for f by the two equations

$$f(\tau+1) = f(\tau)$$
$$f(-1/\tau) = \tau^k f(\tau).$$

Note that $S^2 = (ST)^3 = 1$.

A First Modular Theorem

The function $f_1(\tau) = \eta(\tau)^4$ (recall this is proportional to $\Pi'_h(r)$, and a modular form of weight 2) satisfies

$$f_1(\tau+1) = e^{\pi i/3} f_1(\tau) \text{ and}$$
$$f_1(-1/\tau) = -\tau^2 f_1(\tau).$$

Our first theorem shows that this function is completely characterized by some simple assumptions and a modular argument.

To begin, define any function Π(r) on the positive real axis of the form

$$\Pi(r) = \hat{q}^{\alpha} \sum_{n=1}^{\infty} a_n \hat{q}^n$$

$$(recall\ \hat{q} = e^{\pi i \tau} = \sqrt{q} = e^{-\pi r})$$

to be a conformal block of dimension α (the a_n are assumed to be real). If only the even a_n are non–zero, we call it an even conformal block. The function $f_1(\tau)$ has this form.

Theorem 1. *Let $\Pi(r)$ be any function on the positive real axis such that*

(i) *$\Pi(r)$ is an* even *conformal block with dimension $\alpha > 0$;*

(ii) $\Pi(1/r) = 1 - \Pi(r)$.

Then $\alpha = 1/3$ and $\Pi(r)$ is Cardy's function.

It's easy to prove this. Let f be the analytic continuation of Π'. Using an obvious notation, we have $f\,|_2\, S = -f$, $f\,|\,T = e^{\pi i \alpha} f$. Hence $f\,|\,(ST)^3 = f = -\,e^{3\pi i \alpha} f$ implies that 3α is an odd integer. Therefore f^6 is a (cusp) form of weight 12 (on Γ_1). A standard result in modular forms is that this is a one–dimensional space, spanned by η^{24}.

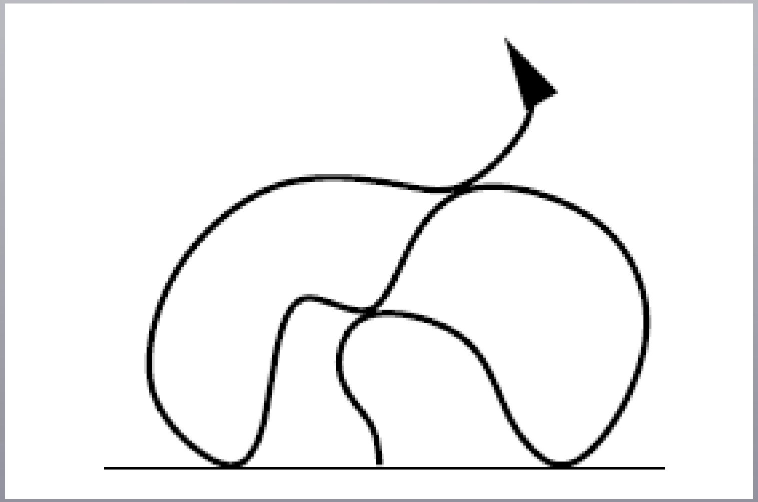
This result is unexpected. Why modular transformations should be relevant in systems with edges is mysterious. (Torii are another matter, of course.) The transformation properties under S follow from duality symmetry, as mentioned. The T property comes from the CFT analysis, and has no obvious simple physical origin.

Crossing in SLEs

Stochastic Löwner Evolution (or Schramm–Löwner Evolution) is a rigorous theory of stochastic conformal maps, driven by a Brownian process of speed κ, $B(\kappa t)$. The real axis, at $t = 0$, is black for $x < 0$ and white for $x > 0$. Thus, in a percolation model, there will be a path γ from $x = 0$ to $x = \infty$ (roughly, this path separates the regions connected to the black part of the boundary from those connected to the white part).

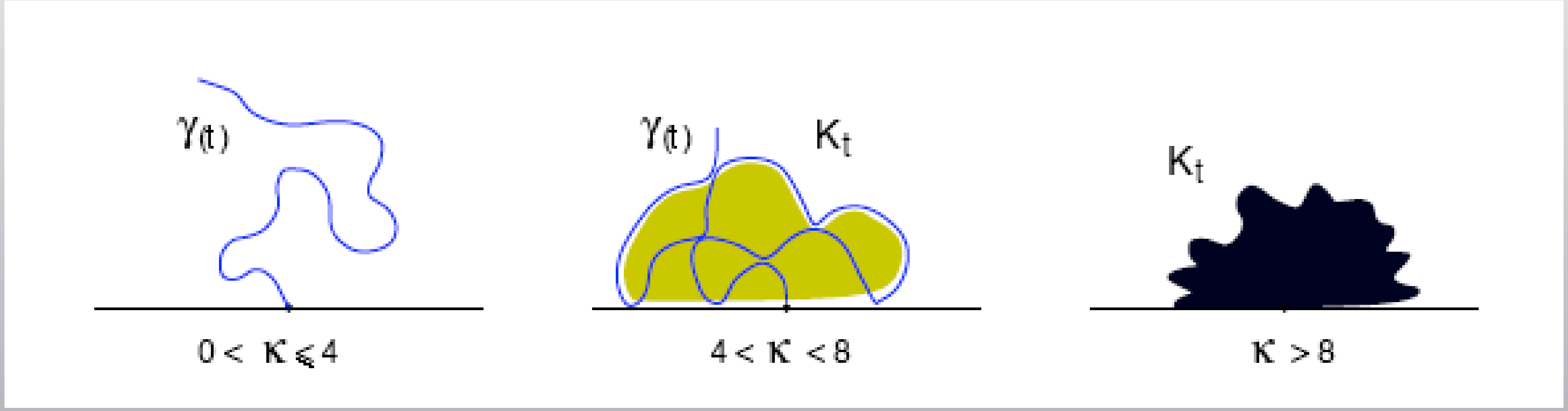
[Schramm, Warner, Lerner, ...]

SLE describes this path in the continuum limit, for a set of models that (roughly speaking) generalize percolation. Furthermore, the path is generated by an exploration process, that is we imagine it to be a function $\gamma(t)$ of time. In SLE, this path is called the trace.



Now in percolation, γ cannot cross itself, but its continuum limit can touch.

So there may be regions that are enclosed by the trace and are separated from infinity by it. The union of such regions and the trace itself, up to time t, is the hull K_t. The nature of the hull depends on the speed κ.

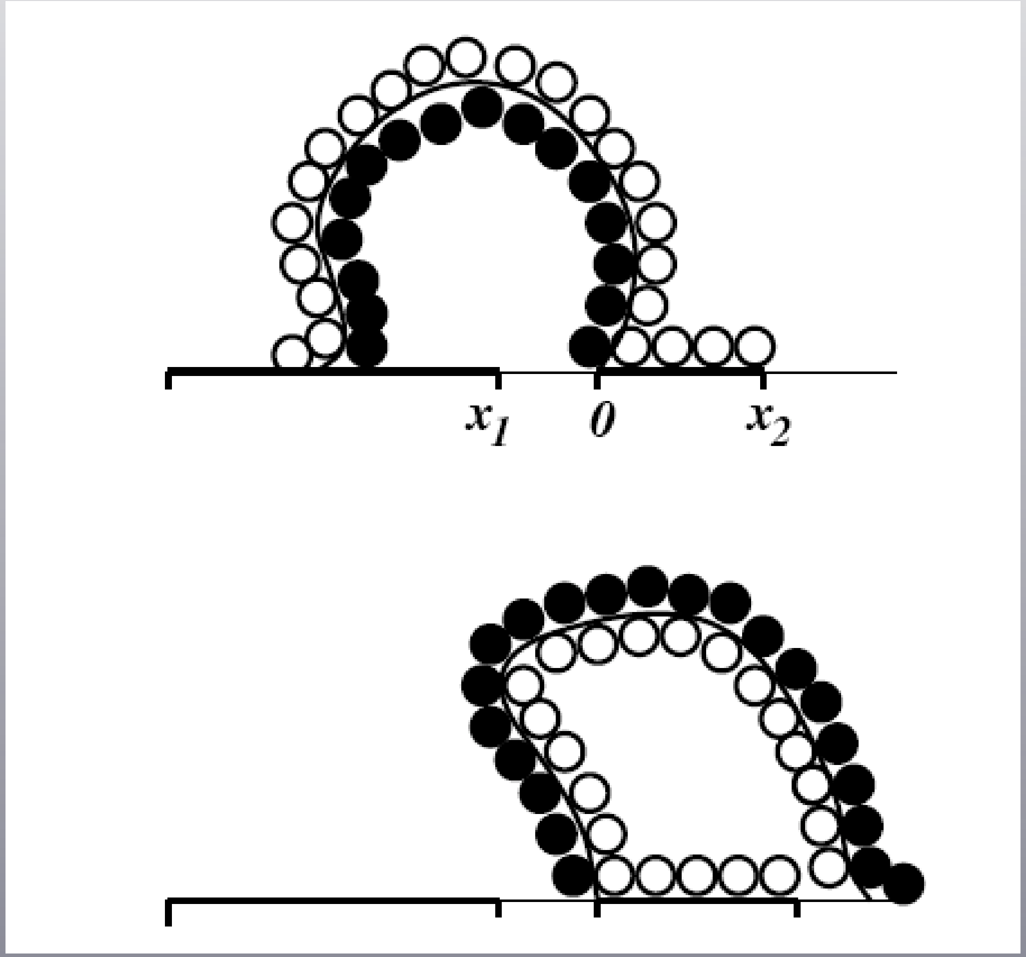


For 0 ≤ κ ≤ 4 the path is simple, for 4 < κ < 8 it can touch without crossing, while for κ ≥ 8 it is space filling. We will see that the limits κ = 4 and κ = 8 arise from modular considerations as well.

The basic equation of SLE gives the time–dependent (regularizing) conformal map $g_t(z)$ from the complement of the hull to the upper half plane

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t},$$

The crossing probability is a bit subtle. Whether or not there is a horizontal crossing (on white) depends on which end point is “swallowed” first by $\gamma(t)$:



There is a crossing on white disks from $(-\infty, x_1)$ to $(0, x_2)$ iff x_1 is swallowed by the SLE before x_2. Think Hex!

The corresponding horizontal crossing probability then follows by stochastic calculus (Itô's equation). It is given by a generalization of Cardy's formula, valid for $\kappa > 4$,

$$F(\lambda;\kappa) = \frac{\Gamma(2-8/\kappa)}{\Gamma(1-4/\kappa)\Gamma(2-4/\kappa)}\,\lambda^{1-4/\kappa}\,{}_2F_1\big(1-\frac{4}{\kappa},\frac{4}{\kappa};2-\frac{4}{\kappa};\lambda\big)$$

It is easy to show that $F(\lambda;\kappa)$ satisfies the same duality condition as Cardy's formula, and reduces to it when $\kappa = 6$. Further, the hypergeometric functions involved again satisfy c–a=1, so that one has

$$F(\lambda;\kappa) = \frac{\Gamma(2-8/\kappa)}{\Gamma(1-4/\kappa)^2}\int_0^\lambda \big(t(1-t)\big)^{-4/\kappa}dt$$

Here $\Pi(r)$, the crossing probability on the rectangle, is a conformal block, but no longer even. Thus $\sqrt{q}$ enters to odd powers, so we are forced to work with the θ–group Γ_θ, generated by S and T^2, i.e.

$$f(\tau+2) = f(\tau)$$
$$f(-1/\tau) = \tau^k f(\tau).$$

The other main ingredient in our modular recipe is the standard "sum rule" satisfied by the zeros of a modular form. For Γ_θ this reads

$$\nu_\infty(f) + \nu_1(f) + \frac{1}{2}\nu_i(f) + \sum_{P \in \mathbf{H}/\Gamma_\theta,\, P \neq i} \nu_P(f) = \frac{k}{4}$$

Here k is the weight of the modular form, and ν_z is the order of the zero at point z. (The points z = 1 and z = i are cusps of the fundamental region and therefore weighted differently. For a conformal block, $\nu_\infty = \alpha$.) This leads to

A Second Modular Theorem

Here, the weight k = 2, so the rhs is 1/2, $\nu_\infty = \alpha > 0$, $\nu_1 > 0$ (and real), and ν_i and ν_P are non–negative integers. Hence $\nu_i = \nu_P = 0$, $\nu_1 = 1/2 - \alpha$, and we find

Theorem 2. *Let $\Pi_1(r)$ be any function on the positive real axis such that*

(i′) $\Pi_1(r)$ *is a conformal block of dimension $\alpha \in \mathbb{R}$ with coefficients a_n of polynomial growth;*

(ii) $\Pi_1(1/r) = 1 - \Pi_1(r)$.

Then $0 < \alpha \leq 1/2$ and $\Pi_1(r) = \Pi_h(r;\alpha)$, the generalized Cardy's function $F(\lambda; \kappa)$.

The bounds on α (which arise from the assumption that our function is "holomorphic at the cusps") are equivalent to $4 < \kappa < 8$, which have a meaning in terms of SLEs as mentioned. The assumption of polynomial growth, however, has no obvious physical interpretation.

Finally we have

Theorem 3. *Let α and $\Pi_1(r)$ be as in Theorem 2 and $\Pi_2(r)$ be a second function satisfying*

(iii) $\Pi_2(r) = e^{-\pi\beta r}\sum_{n=0}^{\infty} b_n e^{-\pi n r}$ *for some* $\beta \in \mathbb{R}$, *with* $\{b_n\}$ *of polynomial growth;*

→ (iv) $\Pi_-(1/r) = \Pi_-(r)$, *where* $\Pi_- := \Pi_1 - \Pi_2$.

Then

(a) $0 < \beta \leq 1,\ \beta \neq \alpha$.

(b) *The function* $\Pi_-(r)$ *is given by the formula*

$$\Pi_-(r) \;=\; C(\alpha,\beta)\int_r^\infty \frac{\eta(it)^{20-48\alpha}}{(\eta(it/2)\eta(2it))^{8-24\alpha}}\int_1^t \frac{\eta(iu)^{20-48(\beta-\alpha)}}{(\eta(iu/2)\eta(2iu))^{8-24(\beta-\alpha)}}\,du\,dt\,, \qquad (29)$$

with

$$C(\alpha,\beta) = 2^{4\beta+1}\pi^2\,\frac{\Gamma(2\alpha)\Gamma(2\beta-2\alpha)}{\Gamma(\alpha)^2\Gamma(\beta-\alpha)^2}\,. \qquad (30)$$

(c) *If also* $\Pi_2(r)$ *and* $\Pi_-(r)$ *are positive for all* $r > 0$, *then* $\beta > \alpha$ *and*

$$\Pi_2(r) \;=\; C(\alpha,\beta)\int_r^\infty \frac{\eta(it)^{20-48\alpha}}{(\eta(it/2)\eta(2it))^{8-24\alpha}}\int_t^\infty \frac{\eta(iu)^{20-48(\beta-\alpha)}}{(\eta(iu/2)\eta(2iu))^{8-24(\beta-\alpha)}}\,du\,dt\,. \qquad (31)$$

The functions Π_1, Π_2 and Π_- are meant to be generalizations of Π_h, $\Pi_{h\underline{v}}$ and Π_{hv}, respectively. Thus this theorem gives (up to the undetermined parameter β) a generalization of the horizontal–vertical crossing probability to SLEs (for which there are no known results). The key element is assumption (iv), the rotation by 90^0 symmetry of Π_{hv} mentioned above. The proof is again straightforward, but involved. Setting $A = e^{2\pi i\alpha}$ and $B = e^{2\pi i\beta}$ one lets $v := f_2/f_1$ (where f_1 and f_2 are the analytic continuations of Π_1' and Π_2'). This gives

$$\underline{v\,|_0\, S = 2 - v \text{ and } v\,|\,T^2 = (B/A)v}.$$

Letting $g := v'\, f_1$, one finds that

$$g\,|_4\, S = g \text{ and } g\,|\,T^2 = Bg,$$

so that g is a modular form of weight 4 on Γ_θ (with character). This is the main step, but there are many other details.

Higher–Order Modular Forms

Recall that the derivatives of the two percolation crossing probabilities satisfy

$$\Pi'_h(r) = +\,(1/r^2)\Pi'_h(1/r),$$

$$\Pi'_{hv}(r) = -(1/r^2)\Pi'_{hv}(1/r).$$

Using the notation above and taking analytic continuations, this reads

$$f_1 \,|_2\, S = -f_1,$$
$$(f_1 - f_2) \,|_2\, S = +(f_1 - f_2),$$

i.e. f_1 has non–trivial character (due, for percolation, to the duality symmetry) while $f_1 - f_2$ does not.

Subtracting the two equations gives

$$f_2 \,|_2\, S = f_2 - 2f_1;$$

i.e. the modular operation S acting on f_2 gives a term f_2 and another one proportional to a modular form (f_1 in this case); on the other hand, by assumption (recall that f_2 is a conformal block)

$$f_2 \,|\, T^2 = Bf_2.$$

This example leads us to define a second order modular form of weight k as a holomorphic function $f(\tau)$ such that $f\,|_k(\gamma - 1)$ is a modular form of weight k for all elements γ of the group (rather than vanishing as for an ordinary modular form). A considerable amount of progress has been made by number theorists on the theory of these objects...

[Chinta, Diamantis, O'Sullivan,....]